

# Ray-Singer Torsion, Topological field theories and the Riemann zeta function at $s = 3$ .

by

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**Abstract:** Starting with topological field theories we investigate the Ray-Singer analytic torsion in three dimensions. For the lens Spaces  $L(p; q)$  an explicit analytic continuation of the appropriate zeta functions is constructed and implemented. Among the results obtained are closed formulae for the individual determinants involved, the large  $p$  behaviour of the determinants and the torsion, as well as an infinite set of distinct formulae for  $\zeta(3)$ : the ordinary Riemann zeta function evaluated at  $s = 3$ . The torsion turns out to be trivial for the cases  $L(6, 1)$ ,  $L((10, 3)$  and  $L(12, 5)$  and is, in general, greater than unity for large  $p$  and less than unity for a finite number of  $p$  and  $q$ .

## § 1. Introduction

The torsion studied in this paper has its origins in the 1930's, cf. Franz [1], where it was combinatorially defined and used to distinguish various lens spaces from one another. Given a manifold  $M$  and a representation of its fundamental group  $\pi_1(M)$  in a flat bundle  $E$ , this Reidemeister-Franz torsion is a real number which is defined as a particular product of ratios of volume elements  $V^i$  constructed from the cohomology groups  $H^i(M; E)$ .

Since volume elements are essentially determinants then, for any alternative definition of a determinant, an alternative definition of the torsion can be given. Now if one uses de Rham cohomology to compute  $H^i(M; E)$  then these determinants become determinants of Laplacians  $\Delta_p^E$  on  $p$ -forms with coefficients in  $E$ . But zeta functions for elliptic operators can be used to give finite values to such infinite dimensional determinants and so an analytic definition of the torsion results and this is the analytic torsion of Ray and Singer [2,3,4] given in the 1970's; furthermore this torsion was proved by them to be independent of the Riemannian metric used to define the Laplacian's  $\Delta_p^E$ .

This analytic torsion coincided, for the case of lens spaces, with the combinatorially defined Reidemeister-Franz torsion. Finally Cheeger and Müller [5,6] independently proved that the analytic Ray-Singer torsion coincides with the combinatorial Reidemeister-Franz torsion in all cases.

Infinite dimensional determinants also occur naturally in quantum field theories when computing correlation functions and partition functions. In 1978 Schwarz [7] showed how to construct a quantum field theory on a manifold  $M$  whose partition function is a power of the Ray-Singer torsion on  $M$ .

Schwarz's construction uses an Abelian gauge theory but in three dimensions a non-Abelian gauge theory—the  $SU(2)$  Chern-Simons theory—can be constructed and has deep and important properties established by Witten in 1988: Its partition function is the Witten invariant for the three manifold  $M$  and the correlation functions of Wilson loops give the Jones polynomial invariant for the link determined by the Wilson loops—cf. [8,9]. Finally the weak coupling limit of the partition function is a power of the Ray-Singer torsion.

We shall be concerned here with the special situation of three dimensions and with the case where the three manifold  $M$  is a lens space. In the next two sections we describe the precise setting and the analytic continuation while the final section contains our concluding remarks.

## § 2. Topological field theories, analytic torsion and lens spaces

Quantum field theories of the type alluded to in the previous section are usually referred to as topological quantum field theories or simply topological field theories.

It turns out that more than one topological field theory can be used to give the torsion, for an excellent review of this question cf. Birmingham et al. [10]. For example one can take the action

$$S[\omega] = \int_M \omega_n d\omega_n, \quad \dim M = 2n + 1 \quad (2.1)$$

where  $\omega_n$  is an  $n$ -form. The partition function is then

$$Z[M] = \int \mathcal{D}\omega \mu[\omega] \exp[-S[\omega]] \quad (2.2)$$

$S[\omega]$  has a gauge invariance whereby  $S[\omega] = S[\omega + d\lambda]$  and therefore to define the partition function it is necessary to integrate over only inequivalent field configurations. The measure  $\mathcal{D}\omega \mu[\omega]$  thus contains functional delta functions which constrain the integration and play the role of gauge fixing, together with their associated determinants. This measure can be constructed using, for example, the Batalin-Vilkovisky BRST construction [11,12]. We wish to devote more space here to lens spaces and the computation of their torsion and so we turn to that now.

To define the Ray-Singer torsion, or simply torsion, we take a closed compact Riemannian manifold  $M$  over which we have a flat bundle  $E$ . Let  $M$  have a non-trivial fundamental group  $\pi_1(M)$  which is represented on  $E$ —this latter property arises very naturally in the physical gauge theory context where it corresponds simply to the space of flat connections all of whose content resides in their holonomy—In any case the torsion is then the real number  $T(M, E)$  where

$$\ln T(M, E) = \sum_{q=0}^n (-1)^q q \ln \det \Delta_q^E, \quad n = \dim M \quad (2.3)$$

The metric independence of the torsion requires that we assume, in the above definition, that the cohomology ring  $H^*(M; E)$  is trivial; this means that the Laplacians  $\Delta_q^E$  have empty kernels and so are strictly positive definite. Given this fact one may use zeta functions to define  $\det \Delta_q^E$  in the standard way. Recall that if  $P$  is a positive elliptic differential or pseudo-differential operator with spectrum  $\{\mu_n\}$  and degeneracies  $\Gamma_n$  then its associated zeta function  $\zeta_P(s)$  is a meromorphic function of  $s$ , regular at  $s = 0$ , which is given by

$$\zeta_P(s) = \sum_{\mu_n} \frac{\Gamma_n}{\mu_n^s} \quad (2.4)$$

and its determinant  $\det P$  is defined by

$$\ln \det P = - \left. \frac{d\zeta_P(s)}{ds} \right|_{s=0} \quad (2.5)$$

Using this we have

$$\ln T(M, E) = - \sum_0^n (-1)^q q \left. \frac{d\zeta_{\Delta_q^E}(s)}{ds} \right|_{s=0} \quad (2.6)$$

Next we turn to lens spaces. For general background on lens spaces cf. [13,14] and references therein—briefly, a lens space can be constructed as follows: Take an odd dimensional sphere  $S^{2n-1}$ , considered as a subset of  $\mathbb{C}^n$ , on which a finite cyclic group of rotations  $G$ , say, acts. The quotient  $S^{2n-1}/G$  of the sphere under this action is a lens space. More precisely, suppose that  $G$  is of order  $p$ ,  $(z_1, \dots, z_n) \in \mathbb{C}^n$  and the group action takes the form

$$(z_1, \dots, z_n) \mapsto (\exp(2\pi i q_1/p) z_1, \dots, \exp(2\pi i q_n/p) z_n) \quad (2.7)$$

with  $q_1, \dots, q_n$  integers relatively prime to  $p$

then the quotient  $S^{2n-1}/G$  is a lens space often denoted by  $L(p; q_1, \dots, q_n)$ . A formula for the torsion of these spaces was first worked out by Ray [2]. We wish to focus on the situation that obtains when  $n = 2$  and  $G$  is the group  $\mathbf{Z}_p \equiv \mathbf{Z}/p\mathbf{Z}$ . For the most part we shall deal with the lens space  $L(2; 1, 1)$  which, for simplicity, we shall denote by  $L(p)$ ; we shall also use the notation  $L(p; q)$  to denote the lens space  $L(p; 1, q)$ . In passing we note that when  $p = 2$  we have  $L(2) = \mathbf{R}P^3 \simeq SO(3)$ .

The group action above defines a representation  $V$ , say, of  $\pi_1(L(p))$  and also determines a flat bundle  $F = (V \times S^3)/\mathbf{Z}_p$ , over  $L(p)$ . It is the torsion of this  $F$  over  $L(p)$  with which we are concerned here. Using zeta functions the torsion of these lens spaces is therefore given by

$$\ln T(L(p), F) = - \sum_0^3 (-1)^q q \left. \frac{d\zeta_{\Delta_q^F}(s)}{ds} \right|_{s=0} \quad (2.8)$$

As an aid to the calculation of  $\ln T(L(p), F)$  it is useful to introduce the notation

$$\tau(p, s) = - \sum_0^3 (-1)^q q \zeta_{\Delta_q^F}(s) \quad T(p) = T(L(p), F) \quad (2.9)$$

For  $\tau(p, s)$  itself we now have

$$\tau(p, s) = \zeta_{\Delta_1^F}(s) - 2\zeta_{\Delta_2^F}(s) + 3\zeta_{\Delta_3^F}(s) = 3\zeta_{\Delta_0^F}(s) - \zeta_{\Delta_1^F}(s), \text{ using Poincaré duality (2.10)}$$

Making use of the triviality of the kernels of  $\Delta_q^F$  we further obtain [2] the formula

$$\tau(p, s) = 2\zeta_{d^*d_0}(s) - \zeta_{d^*d_1}(s) \quad (2.11)$$

For the individual zeta functions we denote the eigenvalues and their degeneracies by  $\lambda_n(q, p)$  and  $\Gamma_n(q, p)$  respectively giving the expressions

$$\zeta_{d^*d_0}(s) = \sum_n \frac{\Gamma_n(0, p)}{\lambda_n^s(0, p)}, \quad \zeta_{d^*d_1}(s) = \sum_n \frac{\Gamma_n(1, p)}{\lambda_n^s(1, p)} \quad (2.12)$$

It remains to compute these eigenvalues and degeneracies cf. [15]. The eigenvalues are

$$\lambda_n(0, p) = n(n+2), \quad \lambda_n(1, p) = (n+1)^2, \quad n = 1, 2, \dots \quad (2.13)$$

To calculate the degeneracies is more difficult; we make use of the fact that  $S^3$  is a group manifold and proceed as follows: Consider the Laplacians  $d^*d_q$  on  $S^3$ , and  $d^*d_q^F$  on  $L(p)$  also, if  $\lambda$  is an eigenvalue, denote the corresponding eigenspaces by  $\Lambda_q(\lambda)$  and  $\Lambda_q^F(\lambda)$  respectively. Let

$$v(z) \in \Lambda_q(\lambda), \text{ with } z \in S^3 \subset \mathbb{C}^2, \text{ and } g \in \mathbb{Z}_p, \text{ where } g \equiv \exp[2\pi i j/p], \quad 0 \leq j \leq (p-1) \quad (2.14)$$

The element  $g$  acts on  $v(z)$  to give  $g \cdot v(z)$  where

$$g \cdot v(z) = v(gz) \quad \text{where } gz = (\exp[2\pi i j/p]z_1, \exp[2\pi i j/p]z_2) \quad (2.15)$$

The above definitions allow us to define the projection  $P(\lambda)$  on  $\Lambda_q(\lambda)$  by

$$P(\lambda)v = \frac{1}{p} \sum_{g \in \mathbb{Z}_p} \exp[-2\pi i j/p] g \cdot v \quad (2.16)$$

Evidently  $[P(\lambda), d^*d_q] = 0$  and so  $P(\lambda)$  projects the space  $\Lambda_q(\lambda)$  onto the space  $\Lambda_q^F(\lambda)$ . Finally this means that we obtain a formula for the degeneracy  $\Gamma_n(q, p)$ , namely

$$\Gamma_n(q, p) = \text{tr} \left( P|_{\Lambda_q^F(\lambda)} \right) = \frac{1}{p} \sum_{j=0}^{(p-1)} \exp[-2\pi i j/p] \text{tr} \left( g|_{\Lambda_q^F(\lambda)} \right) \quad (2.17)$$

To actually apply this formula we now add in the fact that  $S^3$  is the group manifold for  $SU(2)$ . The Peter-Weyl theorem tells us, in this case where all representations are self-conjugate, that

$$L^2(S^3) = L^2(SU(2)) = \bigoplus_{\mu} c_{\mu} D_{\mu} = \bigoplus_{\mu} D_{\mu} \otimes D_{\mu} \quad (2.18)$$

where  $c_\mu$  measures the multiplicity of the representation  $\mu$  which must therefore be  $\dim D_\mu$ . But Hodge theory gives us the alternative decomposition

$$L^2(S^3) = \bigoplus_{\lambda} \Lambda_0(\lambda) \quad (2.19)$$

In addition the Casimir operator for  $SU(2)$  is a multiple of the Laplacian and, if the representation label  $\mu$  is taken to be the usual half-integer  $j$ , then we know that this Casimir has eigenvalues  $j(j+1)$ , and also that  $\dim D_j = 2j+1$ . These facts identify the Laplacian  $\Delta_0 = d^*d_0$  as four times the Casimir and identify  $\Lambda_0(\lambda)$  as  $\dim D_j$  copies of  $D_j$ . Thus if we set  $n = 2j$ , so that  $n$  is always integral, then we have the degeneracy formula

$$\Gamma_n(0, p) = \frac{(n+1)}{p} \sum_{j=0}^{(p-1)} \exp[-2\pi i j/p] \chi^{n/2}(2\pi j/p) \quad (2.20)$$

where  $\chi^j(\theta)$  denotes the  $SU(2)$  character, on  $D_j$ , for rotation through the angle  $\theta$ ; i.e.

$$\chi^j(\theta) = \frac{\sin((2j+1)\theta)}{\sin(\theta)} \quad (2.21)$$

Hence our explicit degeneracy formula for 0-forms on  $L(p)$  is

$$\Gamma_n(0, p) = \frac{(n+1)}{p} \sum_{j=0}^{(p-1)} \exp[-2\pi i j/p] \frac{\sin(2\pi(n+1)j/p)}{\sin(2\pi j/p)} \quad (2.22)$$

We now have to find the analogous formula for the 1-forms. The formula that results is

$$\Gamma_n(1, p) = \frac{1}{p} \sum_{j=0}^{(p-1)} \exp[-2\pi i j/p] \left\{ n \chi^{(n+1)/2}(2\pi j/p) + (n+2) \chi^{(n-1)/2}(2\pi j/p) \right\} \quad (2.23)$$

To simplify the notation we introduce the ' $p$ -averaged character'  $\langle \chi^j \rangle_p$  which we define by

$$\langle \chi^j \rangle_p = \frac{1}{p} \sum_{j=0}^{(p-1)} \exp[-2\pi i j/p] \chi^j(2\pi j/p) \quad (2.24)$$

Finally this gives us a concrete expression for  $\tau(p, s)$ , i.e.

$$\begin{aligned} \tau(p, s) &= \sum_n \left\{ \frac{n \langle \chi^{(n+1)/2} \rangle_p + (n+2) \langle \chi^{(n-1)/2} \rangle_p}{(n+1)^{2s}} - \frac{2(n+1) \langle \chi^{n/2} \rangle_p}{\{n(n+2)\}^{2s}} \right\} \\ &= \tau_+(p, s) - \tau_-(p, s) \end{aligned} \quad (2.25)$$

with the obvious definition for  $\tau_+(p, s)$  and  $\tau_-(p, s)$ .

To actually compute the torsion we need to be able to evaluate these  $p$ -averaged characters. This is a somewhat non-trivial combinatorial task and it is necessary to divide  $n$  up into its conjugacy classes mod  $p$  by writing  $n = pk - j$ ,  $k \in \mathbf{Z}$ ,  $j = 0, 1, \dots, (p-1)$ . We eventually discover that

$$\left\langle \chi^{(pk-j)/2} \right\rangle_p = \begin{cases} \begin{cases} k & \text{for } j = 0, 2, \dots, (p-1) \\ k & \text{for } j = 1 \\ (k-1) & \text{for } j = 3, 5, \dots, (p-2) \end{cases} & \text{if } p \text{ is odd} \\ \begin{cases} 0 & \text{for } j = 0, 2, \dots, (p-2) \\ 2k & \text{for } j = 1 \\ (2k-1) & \text{for } j = 3, 5, \dots, (p-1) \end{cases} & \text{if } p \text{ is even} \end{cases} \quad (2.26)$$

We must now construct the analytic continuation of the series for  $\tau(p, s)$ . For the details we refer the reader to [15]. We shall just describe here the  $p = 2$  case.

### § 3. The Analytic Continuation

The series to be continued is

$$\tau(p, s) = \sum_n \left\{ \frac{n \langle \chi^{(n+1)/2} \rangle_p + (n+2) \langle \chi^{(n-1)/2} \rangle_p}{(n+1)^{2s}} - \frac{2(n+1) \langle \chi^{n/2} \rangle_p}{\{n(n+2)\}^{2s}} \right\} \quad (3.1)$$

and it already converges for  $\text{Re } s > 3/2$ ; however a calculation of the torsion requires us to work at  $s = 0$ , hence we see the need for, and the extent of, the continuation.

With  $p = 2$  we have

$$\tau(2, s) = \sum_n \left\{ \frac{2(n+1) \langle \chi^{n/2} \rangle_2}{\{n(n+2)\}^s} - \frac{n \langle \chi^{(n+1)/2} \rangle_2 + (n+2) \langle \chi^{(n-1)/2} \rangle_2}{(n+1)^{2s}} \right\} \quad (3.2)$$

But using 2.26 we find that

$$\begin{aligned} \left\langle \chi^{(n+1)/2} \right\rangle_2 &= \left\langle \chi^{(2k-j+1)/2} \right\rangle_2, \quad (n = 2k - j) \\ &= \begin{cases} 0, & j = 1 \\ 2k + 2, & j = 0 \end{cases} \equiv \begin{cases} 0, & n \text{ odd} \\ 2k + 2, & n \text{ even} \end{cases} \end{aligned} \quad (3.3)$$

Similarly

$$\begin{aligned} \left\langle \chi^{(n-1)/2} \right\rangle_2 &= \left\langle \chi^{(2k-j)/2} \right\rangle_2, \quad (n = 2k - j) \\ &= \begin{cases} 2k, & j = 1 \\ 0, & j = 0 \end{cases} \equiv \begin{cases} (n+1), & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \end{aligned} \quad (3.4)$$

Thus  $\tau(2, s)$  becomes

$$\tau(2, s) = \tau_+(2, s) - \tau_-(2, s) = \sum_{n \text{ odd}} \frac{2(n+1)^2}{\{n(n+2)\}^s} - \sum_{n \text{ even}} \frac{2n(n+2)}{(n+1)^{2s}} \quad (3.5)$$

Setting  $n = (2m-1)$  in  $\tau_+(2, s)$  and  $n = 2m$  in  $\tau_-(2, s)$  we have

$$\begin{aligned} \tau(2, s) &= \sum_{m=1}^{\infty} \frac{8m^2}{(4m^2-1)^s} - \sum_{m=0}^{\infty} \frac{2m(2m+2)}{(2m+1)^{2s}} \\ &= \sum_{m=1}^{\infty} \frac{8m^2}{(4m^2-1)^s} - \sum_{m=0}^{\infty} \frac{1}{(2m+1)^{(2s-2)}} + \sum_{m=0}^{\infty} \frac{1}{(2m+1)^{2s}} \end{aligned} \quad (3.6)$$

Now if  $\zeta(s)$  is the usual Riemann zeta function we can use the fact that

$$\sum_{n=1,3,5,\dots} \frac{1}{n^s} = (1-2^{-s})\zeta(s) \quad (3.7)$$

then we get

$$\tau(2, s) = \sum_{m=1}^{\infty} \frac{8m^2}{(4m^2-1)^s} - 2(1-2^{-(2s-2)})\zeta(2s-2) + 2(1-2^{-2s})\zeta(2s) \quad (3.8)$$

The only term in 3.8 without a well defined continuation is the first term. To this end we define the quantity  $A(m, s)$  by

$$\begin{aligned} A(m, s) &= \frac{4m^2}{(4m^2-1)^s} = \frac{4m^2}{(4m^2)^s} \left(1 - \frac{1}{4m^2}\right)^{-s} = \frac{1}{(4m^2)^{(s-1)}} \left\{1 + \frac{s}{4m^2} + \dots\right\} \\ &= \frac{1}{(4m^2)^{(s-1)}} + \frac{s}{(4m^2)^s} + R(m, s), \quad (\text{def. of } R(m, s)) \end{aligned} \quad (3.9)$$

So that the remainder term  $R(m, s)$  is given by

$$\begin{aligned} R(m, s) &= A(m, s) - \frac{1}{(4m^2)^{(s-1)}} - \frac{s}{(4m^2)^s} \\ &= \frac{4m^2}{(4m^2-1)^s} - \frac{1}{(4m^2)^{(s-1)}} - \frac{s}{(4m^2)^s} \end{aligned} \quad (3.10)$$

The definition of the remainder term is chosen to ensure that

$$|R(m, s)| \leq \frac{(\ln m)^\alpha}{m^2} \quad (3.11)$$

and this has the vital consequence that the operations  $d/ds$  (at  $s=0$ ) and  $\sum_m$  commute when applied to  $R(m, s)$ .

Returning to  $\tau(s, 2)$  itself we have

$$\begin{aligned}\tau(2, s) &= 2 \sum_{m=1}^{\infty} A(m, s) - 2(1 - 2^{-(2s-2)})\zeta(2s-2) + 2(1 - 2^{-2s})\zeta(2s) \\ \Rightarrow \tau(2, s) &= 2 \sum_{m=1}^{\infty} \frac{1}{(4m^2)^{(s-1)}} + 2s \sum_{m=1}^{\infty} \frac{1}{(4m^2)^s} + 2 \sum_{m=1}^{\infty} R(m, s) \\ &\quad - 2(1 - 2^{-(2s-2)})\zeta(2s-2) + 2(1 - 2^{-2s})\zeta(2s)\end{aligned}\tag{3.12}$$

Defining

$$R(s) = \sum_{m=0}^{\infty} R(m, s)\tag{3.13}$$

gives a series for  $R(s)$  which is guaranteed to be convergent and the analytic continuation is now complete; thus we can now take the final step which is to differentiate and obtain the torsion  $T(2)$ . The result that we get is that

$$\ln T(2) = \frac{d\tau(2, 0)}{ds} = 28\zeta'(-2) + 2(1 + \ln 4)\zeta(0) + 2R'(0)\tag{3.14}$$

But it is easy to check that  $\zeta(0) = -1/2$  and  $\zeta'(-2) = -\zeta(3)/4\pi^2$  and by our remark above concerning the motive for our choice of definition for  $R(m, s)$  we have

$$\begin{aligned}R'(0) &= \frac{d}{ds} \sum_m R(m, s)|_{s=0} \\ \Rightarrow R'(0) &= \sum_m \left. \frac{dR(m, s)}{ds} \right|_{s=0} \\ &= \sum_m [4m^2 \{\ln(4m^2) - \ln(4m^2 - 1)\} - 1] = - \sum_m [4m^2 \ln(1 - 1/4m^2) + 1]\end{aligned}\tag{3.15}$$

Hence

$$\ln T(2) = -\frac{7}{\pi^2}\zeta(3) - 1 - 2\ln(2) - 2 \sum_m [4m^2 \ln(1 - 1/4m^2) + 1]\tag{3.16}$$

However the series for  $R'(0)$  can be expressed as a trigonometric integral cf. [15]. In fact we have

$$\sum_{m=1}^{\infty} [4m^2 \ln(1 - 1/4m^2) + 1] = -\frac{1}{2} + \frac{4}{\pi^2} \int_0^{\pi/2} dz z^2 \cot(z)\tag{3.17}$$

which means that

$$\ln T(2) = -\frac{7}{\pi^2}\zeta(3) - 2\ln(2) - \frac{8}{\pi^2} \int_0^{\pi/2} dz z^2 \cot(z)\tag{3.18}$$



This formula 3.18 above for  $T(2)$  can be pushed even further; by using Ray's expression [2] for the torsion we can deduce that

$$\ln T(p) = -\frac{4}{p} \sum_{j=1}^{(p-1)} \sum_{k=1}^p \cos\left(\frac{2jk\pi}{p}\right) \ln\left(2 \sin\left(\frac{2k\pi}{p}\right)\right) \exp\left[\frac{2k\pi i}{p}\right] = -4 \ln\left(2 \sin\left(\frac{\pi}{p}\right)\right) \quad (3.19)$$

which, for  $p = 2$ , becomes simply

$$\ln T(2) = -4 \ln(2) \quad (3.20)$$

Hence we straightaway have that

$$-4 \ln(2) = -\frac{7}{\pi^2} \zeta(3) - 2 \ln(2) - \frac{8}{\pi^2} \int_0^{\pi/2} dz z^2 \cot(z) \quad (3.21)$$

Or

$$\zeta(3) = \frac{2\pi^2}{7} \ln(2) - \frac{8}{7} \int_0^{\pi/2} dz z^2 \cot(z) \quad (3.22)$$

in other words our computation of the torsion has given us a formula for  $\zeta(3)$ .

In [15] we construct the continuation for arbitrary  $p$  but here we limit ourselves to quoting the torsion formula for  $p$  odd which (recall that  $\ln T(p) = -4 \ln(2 \sin(\pi/p))$ ) is

$$\begin{aligned} \ln T(p) = & -\frac{(p^3 - 1)}{p} \frac{\zeta(3)}{\pi^2} - \frac{2}{\pi} (p-2) \int_0^{\frac{\pi}{p}} dz z \cot(z) + \frac{2}{\pi} (p-2) \int_0^{\frac{(p-1)\pi}{p}} dz z \cot(z) \\ & - \frac{2p}{\pi^2} \int_0^{\frac{\pi}{p}} dz z^2 \cot(z) - \frac{2p}{\pi^2} \int_0^{\frac{(p-1)\pi}{p}} dz z^2 \cot(z) - \frac{4 \ln(2 \sin(\frac{\pi}{p}))}{p} \\ & + \frac{16}{\pi} \sum_{l=1}^{(p-3)/2} l \int_0^{\frac{2l\pi}{p}} dz z \cot(z) - \frac{4p}{\pi^2} \sum_{l=1}^{(p-3)/2} \int_0^{\frac{2l\pi}{p}} dz z^2 \cot(z) \\ & - \frac{4}{p} \sum_{l=1}^{(p-3)/2} 4l^2 \ln\left(2 \sin\left(\frac{2l\pi}{p}\right)\right), \quad \text{for } p \text{ odd} \end{aligned} \quad (3.23)$$

#### § 4. Concluding remarks

These formulae have yet to be elucidated further.

A thought provoking fact is that  $\zeta(3)$  occurs in a recent paper of Witten [16] where, after multiplication by a known constant, it gives the volume of the symplectic space of flat connections over a *non-orientable* Riemann surface. The corresponding calculation for *orientable* surfaces (where the volume element is a rational cohomology class) allows a cohomological rederivation of the irrationality of  $\zeta(2)$ ,  $\zeta(4)$ , ... This paper also involves the torsion but in two dimensions rather than three. The proof that  $\zeta(3)$  is irrational was only obtained in 1978 cf. [17] and the rationality of  $\zeta(5)$ ,  $\zeta(7)$ , ... is at present open.

Our technique, applied in five dimensions instead of three would yield formulae for  $\zeta(5)$  but their nature is as yet unclear.

Further interesting results are that the  $T(p)$  is trivial (i.e. unity) when  $p = 6$ ; and that if we work with  $L(p, q)$  rather than  $L(p)$  then the only other three dimensional lens spaces for which the torsion is trivial are  $L(10, 3)$  and  $L(12, 5)$ . The large  $p$  behaviour of the determinants is also computable:  $T(p)$  grows as  $p^4/(2\pi)^4 + p^2/6(2\pi)^2$  for large  $p$ , while the determinants grow much faster than quartically.

### References

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